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THE STEADY MOTIONS OF A DISC ON AN ABSOLUTELY ROUGH PLANE†

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The branching of the steady motions of a heavy circular disc on an absolutely rough horizontal plane is investigated. The motions corresponding to critical points of the energy integral at fixed levels of two other integrals having the form of hypergeometric series are considered. © 2000 Elsevier Science Ltd. All rights reserved.

It is well known that conservative non-holonomic Chaplygin systems with pseudo-cyclical coordinates, in addition to an energy integral, can also have other first integrals, the explicit form of which is unknown. Nevertheless, the existence and stability of the steady motions of such systems can be investigated using the Routh–Lyapunov–Salvadori theory (see, for example, [1–3]. However, it is extremely difficult to investigate the branching of the steady motions of such systems using the Poincaré–Chetayev bifurcation theory and has not yet been carried out since classical bifurcation theory requires a knowledge of the explicit form of all the first integrals.

Below we attempt to construct Poincaré–Chetayev bifurcation diagrams in the problem of the steady motions of a heavy disc on an absolutely rough horizontal plane. As is well known [4–7], in this problem, in addition to the energy integral, there are two integrals corresponding to two pseudocyclical coordinates (the angles of precession and natural rotation) and represented in the form of hypergeometric series.

Consider the motion, without slipping, of a heavy circular disc, resting on a horizontal plane. Suppose m is the mass of the disc, a is the radius, A_1 and A_3 are the equatorial and axial moments of inertia respectively and g is the acceleration due to gravity.

Following the approach described in [3], we will define the position of the disc by the Cartesian coordinates of the projection of its centre onto the horizontal plane and the Euler angles θ, ψ and φ (θ is the angle between the plane of the disc and the reference plane, ψ is the precession angle and φ is the angle of natural rotation). The equations of motion, taking the coupling equations into account, which express the fact that there is no slipping of the disc at its point of contact with the reference plane, can then be written in the form

$$\begin{aligned}
&(A_1+ma^2)\ddot{\theta} = A_1 q^2 \operatorname{ctg}\theta - (A_3+ma^2)qr - mga\cos\theta \\
&(A_3+ma^2)\dot{r} = ma^2 q\dot{\theta} \\
&A_1\frac{d}{dt}(q\sin\theta) = A_3 r\sin\theta\dot{\theta} \\
&q = \dot{\psi}\sin\theta, \quad r = \dot{\varphi}+\dot{\psi}\cos\theta
\end{aligned} \tag{1}$$

Changing in (1) to a new independent variable—the angle θ, we obtain a second-order differential equation for r. By replacing the angle θ by the new independent variable z, defined by the equation (see [4–7]) $\cos\theta = 1-2z$, we obtain the hypergeometric equation [8]

$$z(1-z)\frac{d^2r}{dz^2}+(1-2z)\frac{dr}{dz}-Br=0, \quad B=\frac{ma^2A_3}{A_1(A_3+ma^2)} \tag{2}$$

Hence, the problem of the motion of a disc is integrated using the hypergeometric function $F(\xi, \eta, \zeta; z)$. We recall that the hypergeometric series $F(\xi, \eta, \zeta; z)$ converges uniformly on any section of the numerical axis lying inside the range $-1 < z < 1$ [8].

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For Eq. (2)

$$\zeta = 1, \ \xi + \eta = 1, \ \xi\eta = B$$

and consequently its general solution has the form

$$r = c_1F(\xi, \eta, 1; z) + c_2F(\xi, \eta, 1; 1-z)$$

where c_1 and c_2 are arbitrary constants and ξ and η are the roots of the quadratic equation

$$s^2 - s + B = 0$$

Reverting from z to the previous independent variable θ we obtain

$$r = c_1u_1 + c_2u_2$$
$$u_1 = F\left(\xi, \eta, 1; \sin^2\frac{\theta}{2}\right), \quad u_2 = F\left(\xi, \eta, 1; \cos^2\frac{\theta}{2}\right) \tag{3}$$

Taking into account the expression for the derivative of a hypergeometric function, we obtain from (1) and (3)

$$q = \frac{A_3}{2A_1}\sin\theta\,(c_1\upsilon_1 - c_2\upsilon_2)$$
$$\upsilon_1 = F\left(\xi+1, \eta+1, 2; \sin^2\frac{\theta}{2}\right), \quad \upsilon_2 = F\left(\xi+1, \eta+1, 2; \cos^2\frac{\theta}{2}\right) \tag{4}$$

We know (see [3, 7]), that the disc can execute steady motions of the form

$$\theta = \alpha = \text{const}, \quad \dot\theta = 0, \quad q = q_0 = \text{const}, \quad r = r_0 = \text{const}$$

if the three constants α, q_0 and r_0 satisfy the single equation

$$A_1q_0^2\,\mathrm{ctg}\,\alpha - (A_3 + ma^2)q_0r_0 - mga\cos\alpha = 0 \tag{5}$$

For simplicity we will henceforth assume that $A_1 = kma^2, A_3 = 2\,kma^2$, where $k = 1/2$ in the case of a hoop and $k = 1/4$ in the case of a disc. In addition, we denote the dimensionless constants of the first integrals, specified implicitly by relations (3) and (4), by $X = c_1\sqrt{(a/g)}$ and $Y = c_2\sqrt{(a/g)}$. With this notation, Eq. (5) can be rewritten in the form

$$a_{11}X^2 + 2a_{12}XY + a_{22}Y^2 - \cos\alpha = 0$$
$$a_{ii} = \upsilon_i\sin\alpha(k\upsilon_i\cos\alpha + (-1)^i(2k+1)u_i), \quad i = 1, 2 \tag{6}$$
$$a_{12} = \sin\alpha((k + 1/2)(u_1\upsilon_2 - u_2\upsilon_1) - k\upsilon_1\upsilon_2\cos\alpha)$$

For each $\alpha \neq \pi/2$, Eq. (6) specifies a hyperbola, and when $\alpha = \pi/2$ it specifies a pair of intersecting straight lines $X = Y$ and $X = -Y$, which correspond to two single-parameter subfamilies of steady motions of the disc of the form

$$\theta = \frac{\pi}{2}, \quad \dot\theta = 0, \quad r = 2u_*c_1 = \Omega, \quad q = 0; \quad u_* = F\left(\xi, \eta, 1; \frac{1}{2}\right) \tag{7}$$

$$\theta = \frac{\pi}{2}, \quad \dot\theta = 0, \quad q = 2\upsilon_*c_1 = \omega, \quad r = 0; \quad \upsilon_* = F\left(\xi+1, \eta+1, 2; \frac{1}{2}\right) \tag{8}$$

These subfamilies correspond to uniform rolling of a vertically situated disc along a straight line (7) and uniform rotation of a disc about a vertically situated diameter (8). The first is stable (unstable) when

$$\Omega^2 > \Omega_0^2 = \frac{g}{2a(2k+1)} \quad (\Omega^2 < \Omega_0^2)$$

while the second is stable (unstable) when

$$\omega^2 > \omega_0^2 = \frac{g}{a(k+1)} \quad (\omega^2 < \omega_0^2)$$

(see [3, 7]). These conditions can be rewritten as follows:

$$X^2 > X_+^2 = (8(2k+1)u_*^2)^{-1}$$

$$X^2 > X_-^2 = (4(k+1)v_*^2)^{-1}$$

We will investigate the behaviour of the steady motions of the system close to the value $\theta = \pi/2$. To do this we will consider Eq. (6) when $\alpha = \pi/2 + \varepsilon_1$, $X = X_0 + \varepsilon_2$, $Y = Y_0 + \varepsilon_3$, where X_0 and Y_0 are certain fixed values, while $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are small quantities. Expanding the left-hand side of Eq. (6) in series in $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (we use the hypergeometric series to expand the function F) and retaining terms no higher than first-order infinitessimals, we obtain the relation

$$(2k+1)u_* v_*(Y_0^2 - X_0^2) + 2(2k+1)u_* v_*(Y_0\varepsilon_3 - X_0\varepsilon_2) +$$
$$+\left[1 - \frac{(X_0+Y_0)^2}{4X_+^2} - \frac{(X_0-Y_0)^2}{4X_-^2}\right]\varepsilon_1 = 0$$

When $X_0 = \pm Y_0$ this relation takes the form

$$\left(1 - \frac{X_0^2}{X_\pm^2}\right)\varepsilon_1 = 2(2k+1)u_* v_* X_0(\varepsilon_2 \mp \varepsilon_3)$$

Hence, non-trivial solutions of Eq. (6) exist in the neighbourhood of the families $X = \pm Y$ when

$$\operatorname{sign}\left(\left(X_\pm^2 - X_0^2\right)\varepsilon_1\right) = \operatorname{sign}(X_0(\varepsilon_2 \mp \varepsilon_3))$$

This analysis enables us to construct the surface $\alpha = \alpha(X, Y)$ in the neighbourhood of the straight lines $X = \pm Y$ using Eq. (6).

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REFERENCES

1. MINDLIN, I. M. and POZHARITSKII, G. K., The stability of the steady motions of a heavy solid of revolution on an absolutely rough horizontal plane. *Prikl. Mat. Mekh.*, 1965, **29**, 4, 742–745.
2. KARAPETYAN, A. V., The stability of the steady motions of systems of a certain form. *Izv. Akad. Nauk SSSR. MTT*, 1983, 2, 45–52.
3. KARAPETYAN, A. V., *The Stability of Steady Motions*. Editorial URSS, Moscow, 1998.
4. VORONETS, P. V., *The Equations of Motion of a Solid Rolling Without Slipping Along a Fixed Plane*. Tip. Univ. Sv. Vladimira, Kiev, 1903.
5. MARKEYEV, A. P., *The Dynamics of a Body in Contact with a Solid Surface*. Nauka, Moscow, 1992.
6. APPELL, P., Sur l'intégration des équations du mouvement d'un corps pesant de revolution roulant par une arête circulaire sur un plane horizontal; cas particulier du cerceau. *Rend. circ. mat. di Palermo*. 1900, **14**, 1–6.
7. NEIMARK, Yu. I. and FUFAYEV, N. A., *Dynamics of Non-holonomic Systems*. Nauka, Moscow, 1967.
8. GRADSHTEIN, I. S. and RYZHIK, I. M., *Tables of Integrals, Sums, Series and Products*. Academic Press, New York, 1970.

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